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A Sufficient Condition for the Complete Reducibility of the Regular Representation*

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The "Mackey machine" is heavily employed to prove the following theorem. Let G be a separable locally compact group. Suppose that every positive definite function p on G which vanishes at infinity is associated with the regular representation R, i.e., $p(g) = (R_g\varphi, \varphi)$ for some L^2 function φ. Then R decomposes into a direct sum of irreducible representations. This generalizes the theorem of Figà-Talamanca for unimodular groups. Although we use his result several times, our techniques are basically very different, the most difficult part occurring in a connected Lie group context.

I. INTRODUCTION

The research represented here is an outgrowth of our interest in and investigation of groups whose regular representation decomposes into a direct sum of irreducible representations. We felt that at least the L^2 harmonic analysis for such a group would be considerably easier than in the general case. Indeed the $ax + b$ group is an example, and in [10] some harmonic analysis results were obtained, e.g., the complete continuity of certain convolution kernels. It was our first impression that groups with completely reducible regular representations were rare, so that our first efforts were directed toward the construction of other examples. A number of totally disconnected groups with completely decomposable regular representations have been found by Mauceri and Picardello. (see [14]). These were apparently inspired by the example of Fell, described in Section IV of [1], of a noncompact totally disconnected unimodular group whose entire dual space is countable. We decided to seek some connected examples, and found them relatively easy to uncover. In [3] a variety of connected Lie groups with completely reducible regular representations is exhibited, including groups which are nonamenable and groups which are non-type I.

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In this paper we prove a sufficient condition on a separable group ensuring that its regular representation decomposes into irreducibles. To describe this sufficient condition, let us examine the situation in the Abelian case. The locally compact Abelian groups whose regular representations completely decompose are exactly the compact ones. Another result which distinguishes compact Abelian groups among all commutative groups is the following.

THEOREM 1.1. *A locally compact Abelian group G is compact if and only if the following is true: If p is a positive definite function on G which vanishes at infinity, then p is of the form $p(g) = (R_g\varphi, \varphi) = \int \varphi(yg)\bar{\varphi}(y)\,dy$ for some L^2 function φ.*

It is the "dual" version of this which was proved, in general by Hewitt and Zuckerman in [9], and the heart of the matter first by Menchoff in [15] for the circle group.

THEOREM 1.1′. *A locally compact Abelian group G is compact if and only if the following is true: If μ is a finite Borel measure on the dual group $\hat{G}$ of G, and if the inverse Fourier–Stieltjes transform of μ vanishes at infinity on G, then μ is absolutely continuous with respect to Haar measure on $\hat{G}$.*

It is worth mentioning that a number of analysts pursued Menchoff's work on the circle, obtaining results on the rate of decay for the Fourier coefficients of a singular measure (see, for example, [11, 17]).

Now since measure on $\hat{G}$ or positive definite functions on G correspond to representations of G, we may rephrase Theorem 1.1 in yet another way.

THEOREM 1.1.″ *A locally compact Abelian group G is compact if and only if the following is true: If π is a cyclic unitary representation of G whose matrix elements all vanish at infinity, then π is equivalent to a subrepresentation of the regular representation of G.*

Proof. Let v be a cyclic for π, and define p on G by $p(g) = (\pi_g v, v)$. Then p is a positive definite function which vanishes at infinity. Hence, by Theorem 1.1, $p(g) = (R_g\varphi, \varphi)$ for some L^2 function φ. Hence π is equivalent to the cyclic subrepresentation of the regular representation generated by the vector φ. Q.E.D.

Algebras of matrix elements have been studied extensively (see, for example, 5, 16]).

DEFINITION 1.2. Let G be a locally compact group. A *matrix element* for a unitary representation π of G is a function of the form $g \to (\pi_g v, w)$ for v and w vectors in the space $X(\pi)$ of π. The set of all possible matrix elements for all possible unitary representations of G is called the *Fourier–Stieltjes Algebra* of G

and is denoted by $B(G)$. The subset of $B(G)$ consisting of the elements which vanish at infinity is denoted by $B_0(G)$. Finally, the set of all matrix elements for the regular representation is denoted by $A(G)$ and is called the *Fourier Algebra* of G.

Remark. A number of easy things can be said concerning these definitions. For our purposes it is convenient to know that: Every matrix element for the regular representation vanishes at infinity; if π is a cyclic unitary representation with cyclic vector v, then every matrix element for π vanishes at infinity if the function $g \to (\pi_g v, v)$ vanishes at infinity; every matrix element for a unitary representation π vanishes at infinity if the functions $g \to (\pi_g w, w)$ vanish at infinity for a set of vectors w whose linear span is dense.

With this notation, Theorem 1.1″ can be stated as: A locally compact Abelian group G is compact if and only if $A(G) = B_0(G)$. This, together with the observation that the compact Abelian groups are the only commutative groups with completely reducible regular representations, gives us:

THEOREM 1.3. *Let G be a locally compact Abelian group. Then R^G is completely reducible if and only if $A(G) = B_0(G)$.*

It is the generalization to arbitrary groups of this theorem which we wound up trying to prove. Figà-Talamanca, in [7], has shown that if G is separable and unimodular, then $A(G) = B_0(G)$ implies that R^G is completely reducible. We show here that this implication holds for any separable group whatsoever. Conversely, we have found an example of a connected Lie group G for which R^G is completely reducible but $A(G) \neq B_0(G)$. The details of this example appear in [3], however the group in question is the semidirect product of the plane with the group of all two by two real matrices of positive determinant.

We shall need a slightly more general definition than Definition 1.2 for our proof.

DEFINITION 1.4. Let T be a unitary representation of a locally compact group G. We say that T *vanishes at infinity* if all of its matrix elements vanish at infinity. We say that $A(T) = B_0(T)$ if the following is true: If S is a cyclic unitary representation of G which is weakly contained in T (Fell) and which vanishes at infinity, then S is equivalent to a subrepresentation of T.

The hypothesis of our theorem is a shade weaker than that of Theorem 1.3 (only in the nonamenable case), i.e., we shall assume that $A(R^G) = B_0(R^G)$. We want to point out, since we in fact use it later on, that Figà-Talamanca's proof in [7] gives a somewhat stronger result than he states.

THEOREM 1.5 (Figà-Talamanca). *Let T be a subrepresentation of the regular representation of a separable unimodular group, (projection in $VN(G)$). If $A(T) = B_0(T)$, then T is completely reducible.*

We have the following natural result concerning Definition 1.4.

THEOREM 1.6. *Let T be a unitary representation of a locally compact group G. We have that $A(T) = B_0(T)$ if and only if the following is true: If S is a unitary representation of G which is weakly contained in T and which vanishes at infinity, then S and T are not disjoint, i.e., they have a common subrepresentation.*

This theorem simply does away with certain multiplicity arguments. It is proved by letting V be a maximal common subrepresentation of S and T.

Notation. If V is a unitary representation of a closed subgroup H of a locally compact group G, then we denote by $\mathrm{ind}_H{}^G\, V$ the representation of G induced from V. In our proof many different subgroups occur as well as many different representations, so that this notation is invaluable. If it is perfectly obvious what the groups G and H are, we shall write U^V in place of $\mathrm{ind}_H{}^G\, V$.

There are several equivalent ways of defining "indeed" representation. We shall explicitly state the one we use.

Let V be a unitary representation of a closed subgroup H of a separable locally compact group G. Let γ denote a "regular Borel cross section" of G/H (right cosets) into G (see [12]). Let μ be a "quasi-invariant" Borel measure on G/H, i.e., a sigma-finite measure on G/H for which $\mu(E) = 0$ if and only if $\mu(E \cdot g) = 0$ for all g in G. Write $\rho(s, g)$ for the Radon–Nikodym derivative of the measure $E \to \mu(E \cdot g)$ with respect to the measure μ. The induced representation $\mathrm{ind}_H{}^G\, V$, or simply U^V, acts in the Hilbert space tensor product of $L^2(\mu)$ and $X(V)$. The formula for this representation, acting on an elementary tensor $f \otimes \psi$, is

$$(U_g{}^V(f \otimes \psi), (f \otimes \psi)) = \int_{G/H} [\rho(s, g)]^{1/2} f(s \cdot g)\, \bar{f}(s)\, (V_{[\gamma(s)g[\gamma(s\, g)]^{-1}]}\psi, \psi)\, d\mu(s).$$

II

THEOREM 2.1. *Let G be a separable locally compact group, and denote by R^G its right regular representation. If $A(R^G) = B_0(R^G)$, then R^G is completely reducible.*

Remark. The proof we give for Theorem 2.1 seems to be quite deep. It relies very heavily on Mackey theory, and indeed some of what we use is not always thought of as a part of that theory, although it follows directly from his Akta paper [13]. Out of laziness, as much as anything else, we have stated this theorem for separable groups. It appears hopeful that our proof goes through as it stands for the nonseparable case. Of course we do use the Mackey machine, and we do use Figà-Talamanca's result [7] both of which assume separability. However, the applications made in our proof, for nonseparable groups, are of a considerably more restricted character, (uncountable discrete groups, nonmetrizable compact groups, etc.), so that their proofs might conceivably be made to work in these nonseparable situations. Our main interest came from the connected case, for which the proof below suffices even in the nonseparable case.

We begin then by stating two theorems which summarize the relevant facts from Mackey theory which we shall be using.

Suppose N is a closed normal subgroup of a separable locally compact group G, and that L is an irreducible unitary representation of N. Let H denote the "stability subgroup of G for L," i.e., the set of all g in G such that the representation L^g of N is equivalent to L. (L^g is defined by $L_n{}^g = L_{[gng^{-1}]}$.) We assume that H is a closed subgroup of G, an assumption which is always satisfied if N is of type I. According to Mackey [13], there exists an (essentially unique) irreducible multiplier representation L^* of H which extends L and whose multiplier is the inflation to $H \times H$ of a multiplier ω on $H/N \times H/N$. We say that a unitary representation V of the group extension $(H/N)^{\bar{\omega}}$ is of "class I" if the restriction of V to the compact central subgroup T (the circle group) of $(H/N)^{\bar{\omega}}$ is a multiple of the identity character J, and we denote by V^* the $\bar{\omega}$-representation of H/N corresponding to V: $V_y^* = V_{(1,y)}$. The representation U^J of $(H/N)^{\bar{\omega}}$ induced from J will play an important and frequent role in the proof. It is clearly a class I representation. With this notation we have the following theorems.

THEOREM 2.2. (i) *Let V be a class I representation of $(H/N)^{\bar{\omega}}$. The mapping T^V defined on H by $T_h{}^V = [L^* \otimes (V^* \cdot \pi)]_h$ is a unitary representation of H whose restriction to N is a multiple of L.*

(ii) *If V and W are two class I representations of $(H/N)^{\bar{\omega}}$, then the Banach space of intertwining operators for V and W is isomorphic with the Banach space of intertwining operators for T^V and T^W.*

(iii) *The mapping $V \to T^v$ respects direct integrals, equivalence, and is a one-to-one correspondence between the set of all irreducible class I representations of $(H/N)^{\bar{\omega}}$ and the set of all irreducible unitary representations of H which restrict on N to be a multiple of L.*

THEOREM 2.3. *Let T and S be unitary representations of H which restrict on N to be multiples of L.*

(i) *The induced representation U^T restricts on N to be a representation of N which is concentrated on the orbit of ${}^JL$ in $\hat{N}$. (A representation V of a group N is said to be "concentrated" on a subset A of $\hat{N}$ if it can be represented as a direct integral $V = \int \pi^x \, d\mu(x)$ such that each representation π^x belongs to some element of A. A major theorem about type I groups is that V cannot be concentrated on two disjoint subsets of $\hat{N}$. We actually shall use this result for a vector group N, in which case that theorem follows immediately from Stone's theorem.)*

(ii) *The Banach space of intertwining operators for T and S is isomorphic to the Banach space of intertwining operators for U^T and U^S.*

Throughout the proof below, whenever this "Mackey setup" is before us, we shall apply the above notation without comment. We use as well a number of properties of induced representations, for example, inducing in stages, commutation of inducing with direct integrals, and Fell's "Continuity of Inducing" Theorem.

We give next a proposition having to do with this Mackey setup, and having directly to do with the theorem at hand.

PROPOSITION 2.4. *Let G, N, L, H, ω, and U^J be as in the above development.*

(i) *If U^J is completely reducible, then so is* $\mathrm{ind}_N{}^G L$.

(ii) *Suppose L vanishes at infinity on N and that G/H is discrete. If* $A(\mathrm{ind}_N{}^G L) = B_0(\mathrm{ind}_N{}^G L)$, *then* $A(U^J) = B_0(U^J)$.

Proof. Recall from [2] that $\mathrm{ind}_N{}^H L$ is equivalent to $L^* \otimes [(U^J)^* \circ \pi]$, so that statement (i) follows directly from Theorems 2.2 and 2.3. To prove the second, let V be a unitary representation of $(H/N)^\omega$ which is weakly contained in U^J and which vanishes at infinity. By the "Continuity of Restriction" theorem [6], V is a class I representation. From the basic definitions we see that $L^* \otimes (V^* \circ \pi)$ is weakly contained in $L^* \otimes [(U^J)^* \circ \pi]$, and from the "Continuity of Inducing" theorem we see that $\mathrm{ind}_H{}^G[L^* \otimes (V^* \circ \pi)]$ is weakly contained in $\mathrm{ind}_N{}^G L$.

Now $L^* \otimes (V^* \circ \pi)$ vanishes at infinity on H. For if x belongs to the space of L and x' belongs to the space of V, then

$$\begin{aligned} |([L^* \otimes (V^* \circ \pi)]_h (x \otimes x'), (x \otimes x'))| &= |(L_h^* x, x)|\, |(V_{\pi(h)} x', x')| \\ &= |(L_n L_{p(s)}^* x, x)|\, |(V_s x', x')|, \end{aligned}$$

where p denotes a regular cross section of H/N into H. Clearly this expression tends to zero as s approaches infinity by assumption on V. If s then is restricted to a compact set, then the expression tends to zero as n approaches infinity by assumption on L. Hence $L^* \otimes (V^* \circ \pi)$ vanishes at infinity. Because G/H is discrete, we may employ Lemma A below to conclude that $\mathrm{ind}_H{}^G[L^* \otimes (V^* \circ \pi)]$ vanishes at infinity on G. Now by our assumption in (ii), there exists a nonzero intertwining operator for $\mathrm{ind}_H{}^G[L^* \otimes (V^* \circ \pi)]$ and $\mathrm{ind}_H{}^G[L^* \otimes [(U^J)^* \circ \pi]]$, and so by Theorems 2.2 and 2.3 there exists a nonzero intertwining operator for V and U^J. This shows that $A(U^J) = B_0(U^J)$. Q.E.D.

Remark. The fact that $\mathrm{ind}_N{}^H L$ is equivalent to $L^* \otimes [(U^J)^* \circ \pi]$ will be needed later on in a slightly more general setting. Indeed the "irreducibility" assumption on L is quite unnecessary. The proof given in [2] actually gives:

PROPOSITION 2.5. *Let P be a unitary representation of a closed normal subgroup N of a separable locally compact group H. Suppose that for each h in H the representation P^h on N defined by $P_n{}^h = P_{[hnh^{-1}]}$ is equivalent to P, and suppose* *that there exists a multiplier representation P^* of H which extends P and whose multiplier is the inflation to all of $H \times H$ of a multiplier ω on $H/N \times H/N$. Then* $\mathrm{ind}_N{}^H P$ *is equivalent to* $P^* \otimes [(U^J)^* \circ \pi]$.

We present in the next theorem a proof of Theorem 2.1 in case G is almost connected.

THEOREM 2.6. *Let G be an almost-connected separable locally compact group. If $A(R^G) = B_0(R^G)$, then R^G is completely reducible.*

Proof. Let K be a compact normal subgroup of G for which G/K is a Lie group. Because R^G is a direct sum of representations of the form $\mathrm{ind}_K{}^G \varphi$, for φ an irreducible representation of K, it will suffice to show that each such representation $\mathrm{ind}_K{}^G \varphi$ is completely reducible. Fix such a φ and let H denote the stability subgroup of G for φ. Because φ is a discrete point in $\hat{K}$, it follows that H/K must contain the connected component of the identity in the Lie group G/K. Hence H is open. By Proposition 2.4, Theorem 2.6 will be established if we can show the following result for separable Lie groups.

THEOREM 2.6.′ *Let G be a Lie group with a finite number of components, let T be a central one-dimensional torus, and let J be the identity character of T. If* $A(\mathrm{ind}_T{}^G J) = B_0(\mathrm{ind}_T{}^G J)$, *then* $\mathrm{ind}_T{}^G J$ *is completely reducible.*

Note that if P is an irreducible representation of the connected component of the identity G_0 in G, then $\mathrm{ind}_{G_0}^G P$ must be a finite direct sum of irreducible representations of G. (This follows straight from the formulas for induced representations.) Hence it will suffice to show that $\mathrm{ind}_T^{G_0} J$ is completely reducible. We claim that $A(\mathrm{ind}_T^{G_0} J) = B_0(\mathrm{ind}_T^{G_0} J)$. Indeed if V vanishes at infinity and is weakly contained in $\mathrm{ind}_T^{G_0} J$, then by Lemma A we have that $\mathrm{ind}_{G_0}^G V$ vanishes at infinity on G. Of course it is weakly contained in $\mathrm{ind}_T{}^G J$. Consequently $\mathrm{ind}_{G_0}^G V$ is not disjoint from $\mathrm{ind}_T{}^G J$. Because $[\mathrm{ind}_{G_0}^G V] |_{[G_0]}$ has a subrepresentation which is equivalent to V, (the space of functions from $G/[G_0]$ into the space of V which are concentrated on the coset G_0), it follows that $[\mathrm{ind}_T{}^G J] |_{[G_0]}$ is not disjoint from V. Since T belongs to the center of G, we have from Mackey's "Subgroup Theorem" [12] that $[\mathrm{ind}_T{}^G J] |_{[G_0]}$ is a multiple of $\mathrm{ind}_T^{G_0} J$, and therefore V is not disjoint from the representation $\mathrm{ind}_T^{G_0} J$. Consequently Theorem 2.6′ will follow from:

THEOREM 2.6.″ *Let G be a connected Lie group, T a central one-dimensional torus in G, and J the identity character of T. If* $A(\mathrm{ind}_T{}^G J) = B_0(\mathrm{ind}_T{}^G J)$, then $\mathrm{ind}_T{}^G J$ *is completely reducible.*

We prove this by induction on the dimension of G. Before beginning this argument, we make the following observation.

The center of G contains no infinite discrete cyclic subgroups. Indeed if Z

were such a subgroup, let V denote a unitary representation of Z which vanishes at infinity and which is disjoint from the regular representation R^Z of Z. Now T intersects Z only at the identity. So $\mathrm{ind}_T{}^G\, J$, which is equivalent to $\mathrm{ind}_{TZ}^G[\mathrm{ind}_T^{TZ}\, J]$, is equivalent to $\mathrm{ind}_{TZ}^G[J \times R^Z]$. ("$x$" denoting outer Kronecker product here.) Therefore $\mathrm{ind}_T{}^G\, J$ weakly contains $\mathrm{ind}_{TZ}^G[J \times V]$. By Lemma B below, $\mathrm{ind}_{TZ}^G[J \times V]$ vanishes at infinity on G, and by Lemma C it is disjoint from $\mathrm{ind}_{TZ}^G[J \times R^Z]$, i.e., disjoint from $\mathrm{ind}_T{}^G\, J$. But this is a contradiction to our assumption

Now if $\dim G \leqslant 2$, then G must be Abelian. If it is a torus, then the theorem holds since G is compact. If it is a cylinder ($T \times R$), then there is an infinite discrete central subgroup which we have just seen is impossible. Therefore assume that $\dim G \geqslant 3$.

1. We may as well assume that T is the maximum compact normal connected subgroup of G. Indeed if K denotes the maximum compact normal connected subgroup of G, then since $\mathrm{ind}_T{}^G\, J$ is equivalent to $\mathrm{ind}_K{}^G[\mathrm{ind}_T{}^K\, J]$, it will suffice to prove that $\mathrm{ind}_K{}^G\, \varphi$ is completely reducible for each irreducible subrepresentation φ of $\mathrm{ind}_T{}^K\, J$. Just as in the paragraph preceding Theorem 2.6′, we have that the stability subgroup of G for φ is open, and by Proposition 2.4 again $\mathrm{ind}_K{}^G\, \varphi$ is completely reducible if U^J is completely reducible, where U^J is the representation of $(H/K)^{\varpi}$ induced from the identity character J of the central one-dimensional torus T of that group. If $\dim K > 1$, then $\dim((H/K)^{\varpi}) <$ $\dim G$, and U^J is completely reducible by the inductive hypothesis. Hence we may assume that $T = K$.

2. Next, assume G contains a closed vector normal subgroup of positive dimension. We let N denote a minimal closed normal vector subgroup with positive dimension in G. For any nonzero element χ in $\hat{N}$ we have, [4], that the representation $\mathrm{ind}_N{}^G\, \chi$ vanishes at infinity modulo its kernel Q. By the minimality of the subgroup N we have that the connected component of the identity in Q, (a priori a normal subgroup of G and contained in N), is zero dimensional. Therefore Q must be discrete. But discrete normal subgroups of connection groups belong to the center. Hence Q is trivial, and $\mathrm{ind}_N{}^G\, \chi$ vanishes at infinity.

Let us examine the representation $\mathrm{ind}_{NT}^G[\chi \times J]$. This is a subrepresentation of $\mathrm{ind}_{NT}^G[\chi \times R^T]$, which is equivalent to $\mathrm{ind}_N{}^G\, \chi$. Hence $\mathrm{ind}_{NT}^G[\chi \times J]$ vanishes at infinity. But $\mathrm{ind}_{NT}^G[\chi \times J]$ is weakly contained in $\mathrm{ind}_{NT}^G[R^N \times J]$ which is equivalent to $\mathrm{ind}_T{}^G\, J$. Therefore, by assumption on $\mathrm{ind}_T{}^G\, J$, the representation $\mathrm{ind}_{NT}^G[\chi \times J]$ is not disjoint from $\mathrm{ind}_T{}^G\, J$ and is therefore not disjoint from the regular representation R^G of G. This implies that the orbit θ of χ in $\hat{N}$ has positive Lebesgue measure. For otherwise

$$R^G = \mathrm{ind}_N{}^G[R^N] = \mathrm{ind}_N{}^G\left[\int_{[\hat{N}-\theta]} \psi\, d\psi\right] = \int_{[\hat{N}-\theta]} [\mathrm{ind}_N{}^G\, \psi]\, d\psi.$$

Directly from the definitions for induced representations we see that $R^G\,|_N$ is concentrated on $\hat{N} - \theta$. On the other hand, $\operatorname{ind}_{NT}^G[\chi \times J]$, being a subrepresentation of $\operatorname{ind}_N{}^G \chi$, restricts on N (Theorem 2.3) to a representation which is concentrated on θ. Hence $R^G|_N$ and $[\operatorname{ind}_{NT}^G[\chi \times J]]|_N$ are disjoint while R^G and $\operatorname{ind}_{NT}^G[\chi \times J]$ are not disjoint. Since this is impossible, θ has positive Lebesgue measure.

Since this orbit is an analytic submanifold of the Euclidean space $\hat{N}$, and since this orbit has positive measure, it follows that θ is an open submanifold. We have shown that the orbit of any nonzero character of N is open. There are then at most a countable number of orbits. (If $\dim N > 1$ there is but one nontrivial orbit while if $\dim N = 1$ there are two nontrivial orbits.)

If θ is a nontrivial orbit, then $\operatorname{ind}_{NT}^G [\int_\theta \psi \, d\psi \times J]$, which is equivalent to $\int_\theta \operatorname{Ind}_{NT}^G[\psi \times J] \, d\psi$, is, (by Mackey theory as applied to the normal subgroup NT), a multiple of $\operatorname{ind}_{NT}^G[\chi \times J]$ for any element χ in the orbit θ. Therefore it will suffice to show that each of the representations $\operatorname{ind}_{NT}^G[\chi \times J]$ is completely reducible.

Let H denote the stability subgroup of G for the character $\chi \times J$ of the subgroup NT. We may not employ Proposition 2.4 in this situation since G/H need not be discrete nor does $\chi \times J$ vanish at infinity. The argument we give here is in the same spirit as Proposition 2.4 but much more complicated.

We have that $\operatorname{ind}_{NT}^G[\chi \times J]$ is equivalent to $\operatorname{ind}_H{}^G[\operatorname{ind}_{NT}^G[\chi \times J]]$, which is equivalent to $\operatorname{ind}_H{}^G[[\chi \times J]^* \otimes [(U'^J)^* \circ \pi]]$, and by Theorems 2.2 and 2.3 it is enough to show that U'^J is completely reducible. Now the group $(H/NT)^\varpi$, of which U'^J is a representation, has dimension at most $\dim G - 2$. So by the inductive hypothesis, U'^J will be completely reducible if we can show that $A(U'^J) = B_0(U'^J)$.

Thus let V be a unitary representation of $(H/NT)^\varpi$ which vanishes at infinity and is weakly contained in U'^J. It follows directly from the definition that $[\chi \times J]^* \otimes [V^* \circ \pi]$ is weakly contained in $[\chi \times J]^* \otimes [(U'^J)^* \circ \pi]$, and therefore that $\operatorname{ind}_H{}^G[[\chi \times J]^* \otimes [V^* \circ \pi]]$ is weakly contained in $\operatorname{ind}_H{}^G[[\chi \times J]^* \otimes [(U'^J)^* \circ \pi]]$ which is $\operatorname{ind}_{NT}^G[\chi \times J]$.

From Lemma D below it follows that $\operatorname{ind}_H{}^G[[\chi \times J]^* \otimes [V^* \circ \pi]$ vanishes at infinity on G, and so by assumption is not disjoint from $\operatorname{ind}_{NT}^G[\chi \times J]$. On the other hand $[\operatorname{ind}_H{}^G[[\chi \times J]^* \otimes [V^* \circ \pi]]|_{NT}$ must be, by Theorem 2.3, concentrated on the orbit in $(NT)^\wedge$ containing the character $\chi \times J$. Whence $\operatorname{ind}_H{}^G[[\chi \times J]^* \otimes [V^* \circ \pi]]$ is disjoint from any representation $\operatorname{ind}_{NT}^G[\psi \times J]$ for ψ not in θ. Hence $\operatorname{ind}_H{}^G[[\chi \times J]^* \otimes [V^* \circ \pi]]$ is not disjoint from $\operatorname{ind}_H{}^G$ $[[\chi \times J]^* \otimes [(U'^J)^* \circ \pi]$, and therefore, by Theorems 2.2 and 2.3, V is not disjoint from U'^J. We have shown that $A(U'^J) = B_0(U'^J)$, and so by the inductive hypothesis U'^J is completely reducible. Therefore the theorem is proved in this case.

3. Finally assume that G contains no closed vector normal subgroups of positive dimension. If G/T is semisimple, then G is unimodular, and the theorem

follows from [7]. Assume then that G/T is not semisimple, let M denote a closed connected normal Abelian subgroup of G/T, and let M' denote its inverse image in G. Now M can contain no compact part, for otherwise G would contain a compact connected normal subgroup strictly larger than T. Hence M is a vector group.

The center of M' is itself a normal subgroup of G. If it were larger than T, then G would contain a closed vector normal subgroup, and the theorem has been proved in that case. Hence we may assume that M' is a two-step, connected nilpotent Lie group with a one-dimensional torus for its center. Lemma E below deals with such groups. We have that $\mathrm{ind}_T{}^G\, J$ is equivalent to $\mathrm{ind}_{M'}^G\, [\mathrm{ind}_G^{M'}\, J]$, so that $\mathrm{ind}_T{}^G\, J$ is a multiple of $\mathrm{ind}_{M'}^G\, W$, where W is the unique irreducible unitary representation of M' whose restriction to T is a multiple of J. We must show that $\mathrm{ind}_{M'}^G\, W$ is completely reducible. But this now follows by applying Proposition 2.4(ii), then the inductive hypotheses, and finally Proposition 2.4(i).

This completes the proof to Theorem 2.6″ and so also the proof of Theorem 2.6.

We give in the next theorem another special case of Theorem 2.1.

THEOREM 2.7. *Let G be a separable locally compact group, K a compact open subgroup, T a one-dimensional central torus in G, and J the identity character of T. If $A(\mathrm{ind}_T{}^G\, J) = B_0(\mathrm{ind}_T{}^G\, J)$, then $\mathrm{ind}_T{}^G\, J$ is completely reducible.*

Proof. Clearly we may assume that T is contained in K. Denote by D the kernel of the modular function of G. Since the modular function is identically one on every compact subgroup of G, we have that D is open in G. Also D is normal, and we have, using the subgroup theorem just as in the first paragraph of the proof to Theorem 2.6′, that $A(\mathrm{ind}_T{}^D\, J) = B_0(\mathrm{ind}_T{}^D\, J)$. But now D is unimodular, and so $\mathrm{ind}_T{}^D\, J$ is completely reducible by [7]. To show that $\mathrm{ind}_T{}^G\, J$ is completely reducible, we need only verify that $\mathrm{ind}_D{}^G\, P$ is completely reducible for every irreducible subrepresentation of $\mathrm{ind}_T{}^D\, J$. But this now follows by an application of Proposition 2.4(ii), Figà-Talamanca's result applied to the unimodular group $(H/D)^{\varpi}$, and then Proposition 2.4(i).

This completes the proof to Theorem 2.7.

Now at last let us prove Theorem 2.1. Let G_0 denote the connected component of the identity in G and let G_1 be an almost connected open subgroup of G. We begin by showing that R^{G_0} is completely reducible. If L is a compact normal subgroup of G_1 such that G_1/L is a Lie group, then since G_0L is open in G_1, we may as well assume that $G_1 = G_0L$. Let K denote the intersection of L and G_0. Then K is a compact normal subgroup of G_1 and our claim will be established if we show that $\mathrm{ind}_K^{G_0}\, \varphi$ is completely reducible for each φ in $\hat{K}$. Fix such a φ. The orbit in $\hat{K}$ under the action of the compact group L must be finite. Hence the stability subgroup of L for φ is open in L, and we may as wel take L equal to that stability subgroup. (This entails a change of the group G of course.) We let ρ be a multiplier representation of L satisfying $\rho_y \varphi_k(\rho_y^{-1}) = \varphi_{[yky^{-1}]}$ for all k in K and y in L. Let γ be a cross section of L/K into L, an denote by P the representation $\operatorname{ind}_K^{G_0} \varphi$. By the proof to Proposition 2.4 we kno that P is equivalent to $\varphi^* \otimes [(U^J)^* \circ \pi]$. Define a mapping P^* on all of G_0L which is G_1, by $P^*_{[G_0\gamma(s)]} = P_{(g_0)}[\rho_{\gamma(s)} \otimes I]$, where I denotes the identity operato on the space of U^J. We have that P^* is a multiplier representation of G_1 whic extends P. By Proposition 2.5, we have that $\operatorname{ind}_K^{G_1} \varphi$ is equivalent to $P^* \otimes$ $[(U'^J)^* \circ \pi]$.

Now again using the subgroup theorem of Mackey, we have that $A(R^{G_1}) = B_0(R^{G_1})$. Hence by Theorem 2.6, R^{G_1} is completely reducible. So, then, is $\operatorname{ind}_K^{G_1} \varphi$ Therefore P^* is completely reducible. Finally this implies that P itself is completely reducible, and our claim that R^{G_0} is completely reducible is verified

So to complete the proof to Theorem 2.1 we need to show that $\operatorname{ind}_{G_0}^G S$ i completely reducible for each irreducible subrepresentation S of R^{G_0}. But thi now follows by an application of Proposition 2.4, Theorem 2.7, and Propositio 2.4 once again.

The proof is now complete.

LEMMA A. *Let H be an open subgroup of a locally compact group G, and let be a unitary representation of H which vanishes at infinity. Then the induce representation* $\operatorname{ind}_H{}^G S$ *vanishes at infinity.*

Proof. Fix an element ψ in the space of S and let f be the characteristi function of the point H in G/H. The vectors $f \otimes \psi$, for ψ ranging over the spac of S, form a cyclic family of vectors for the representation $\operatorname{ind}_H{}^G S$. Finally let be a cross section of G/H into G. Then

$$|(U_g{}^S(f \otimes \psi), (f \otimes \psi))| = |(U^S_{[h\gamma(y)]}(f \otimes \psi), (f \otimes \psi))| = |f(y)|\ |(S_h\psi, \psi)|,$$

and this clearly tends to zero as $h\gamma(y)$ tends to infinity. Q.E.D

LEMMA B. *Let V be a unitary representation of a closed central subgroup Z of locally compact group G. If V vanishes at infinity on Z, then* $\operatorname{ind}_Z{}^G V$ *vanishe at infinity on G.*

Proof. Let γ be a cross section of G/Z into G, let ψ be a vector in the spac of V, and let f be the characteristic function of a compact set C of G/Z. The

$$\begin{aligned} &|(U^V_{[z\gamma(y)]}(f \otimes \psi), (f \otimes \psi))| \\ &\quad = \left| \int_{G/Z} f(y'y)\bar{f}(y')(V_{[\gamma(y')z\gamma(y)[\gamma(y'y)]^{-1}}\psi, \psi)\, dy' \right| \\ &\quad \leqslant \int_C |f(y'y)|\ |(V_z V_{[\gamma(y')\gamma(y)[\gamma(y'y)]^{-1}]}\psi, \psi)|\, dy'. \end{aligned}$$

Now unless y belongs to the compact set $C^{-1}C$, this integrand is zero. If y is restricted to that compact set, then the elements $\gamma(y')\,\gamma(y)\,[\gamma(y'y)]^{-1}$ belong to a compact subset of Z, and the integral tends to zero as z approaches infinity by assumption on V. Q.E.D.

This lemma seems to be related to some of the results in [8].

LEMMA C. *If V and W are disjoint representations of a closed central subgroup Z of a locally compact group G, then the induced representations U^V and U^W are also disjoint.*

Proof. This follows immediately because $U^V|_Z$ is a multiple of V, if Z is central.

LEMMA D. *Let N be a connected closed normal Abelian subgroup of a connected Lie group G. Suppose β is a character of N, H is the stability subgroup of G for β, β^* is the Mackey extension of β to a multiplier representation of H with multiplier inflated from a multiplier ω on $H/N \times H/N$, and V is a classe I representation of $(H/N)^{\bar{\omega}}$. Assume that the homogeneous space G/H is homeomorphic with a connected component of some punctured Euclidean space $R^j - [0]$, and that the action of G on G/H is by linear transformations. Then if V vanishes at infinity, the induced representation $\operatorname{ind}_H{}^G[\beta^* \otimes (V^* \circ \pi)]$ vanishes at infinity modulo its kernel.*

Proof. Lebesgue measure μ on G/H is quasi-invariant under the action of G since that action is by linear transformation. In fact $\int_{G/H} f(s \cdot g)\,ds = [1/d(g)] \int_{G/H} f(s)\,ds$, where $d(g)$ is the absolute value of the determinant of the linear transformation of R^j determined by g. If T denotes the representation $\beta^* \otimes (V^* \circ \pi)$, then the representation U^T acts in the Hilbert space $L^2(G/H) \otimes X(T)$. If p denotes a cross section of G/H into G, f is an element of $L^2(G/H)$, and ψ is a vector in the space of T (the space of V), and if $g = hp(t)$, then

$$(U_g{}^T(f \otimes \psi), (f \otimes \psi)) = \int_{G/H} [d(g)]^{1/2} f(s \cdot g)\,\bar{f}(s)\,(T_{[p(s)g[p(s\cdot g)]^{-1}]}\psi, \psi)\,ds$$

and we must show that this function of g vanishes at infinity. Clearly it will suffice to do this when f is the characteristic function of a compact cube C in G/H with corner s' and sides $s' + v_1$, $s' + v_2$,..., $s' + v_j$, where the $[v_i]$ form a basis for R^j and in fact v_1 is a positive multiple λ of the point s_0 corresponding to the coset H. We fix such an f, a vector ψ in $X(T)$, and a positive number ϵ. Write E for $|(U_g{}^T(f \otimes \psi), (f \otimes \psi))|$. Then

$$E \leqslant [d(g)]^{1/2} \int_{G/H} f(s \cdot g) f(s)\,ds = [d(g)]^{1/2}\,\mu(C \cap (C \cdot g^{-1})).$$

Consequently $E \leqslant [d(g)]^{1/2}\,\mu(C)$, and $E \leqslant [d(g)]^{1/2}\,\mu(C \cdot g^{-1})$ which is $[d(g)]^{1/2}$

$\mu(C)$. So $E < \epsilon$ unless $d(g)$ lies in a closed interval $[\delta, M]$ of positive real numbers.

Observe next that $v_1 \cdot = (\lambda s_0) \cdot g = \lambda(s_0 \cdot g) = \lambda(s_0 \cdot h \cdot p(t)) = \lambda(s_0 \cdot p(t)) = \lambda t$. So the transform of the cube C by g, i.e., $C \cdot g$, is a parallelepiped one of whose sides is of length $\| \lambda t \|$. Now

$$E \leqslant [d(g)]^{1/2} \int_{G/H} f(s \cdot g) f(s)\, ds$$

$$= [d(g)]^{-1/2} \int_{G/H} f(s) f(s \cdot g^{-1})\, ds \leqslant [\delta]^{-1/2} \mu(C \cap C \cdot g).$$

If $\| t \|$ approaches 0, then the parallelepiped $C \cdot g$ has one side whose length tends to zero. Therefore the measure of the intersection of C with $C \cdot g$ tends to zero by a simple geometric argument. So $E < \epsilon$ unless $\| t \| \geqslant \delta' > 0$. Also, if $\| t \|$ tends to infinity, then the length of one side of $C \cdot g$ tends to infinity. However the total volume of $C \cdot g$ is bounded by $1/\delta$, and so again by a simple geometric argument we see that the measure of $C \cap C \cdot g$ tends to zero. Hence $E < \epsilon$ unless $\| t \| \leqslant M'$. We have then that $E < \epsilon$ unless $\| t \|$ belongs to a closed interval I of positive real numbers.

Now let γ be a cross-section of H/N into H. Then

$$E \leqslant [d(g)]^{1/2} \int_{[C \cap C \cdot g^{-1}]} |(T_{[p(s) n\gamma(y) p(t) [p(s \cdot g)]^{-1}]}\psi, \psi)|\, ds$$

$$\leqslant M^{1/2} \mu(C) \mid \beta^*(p(s)\, n\gamma(y)\, p(t)\, [p(s \cdot g)]^{-1})|$$

$$\times |(V^*_{\pi(p(s) n\gamma(y) p(t) [p(s \cdot g)]^{-1})}\psi, \psi)|$$

$$= M^{1/2} \mu(C) |(V^*_{\pi(p(s) n [p(s)]^{-1})} V^*_{\pi(p(s)\gamma(y) p(t) [p(s \cdot g)]^{-1})}\psi, \psi)|$$

$$= M^{1/2} \mu(C) |(V^*_{\pi(p(s)\gamma(y) p(t) [p(s \cdot g)]^{-1})}\psi, \psi)|.$$

By assumption on V there exists a compact subset K of H/N so that $|(V^*_{\pi(h)}\psi, \psi)| < \epsilon/[M^{1/2} \mu(C)]$ unless $\pi(h)$ belongs to K. But

$$E \leqslant M^{1/2} \int_{[C \cap C \cdot g^{-1}]} |(V^*_{\pi(p(s)\gamma(y) p(t) [p(s \cdot g)]^{-1})}\psi, \psi)|\, ds$$

which then is less than ϵ if $\pi(p(s)\ \gamma(y)\ p(t)\ [p(s \cdot g)]^{-1}$ does not belong to K. But since s is in $C \cap C \cdot g^{-1}$, this does not belong to K if y does not belong to the compact set $K' = \pi[H \cap [[\overline{p(C)}]^{-1}\, \overline{\gamma(K)}\, \overline{p(C)}\, [\overline{p(I)}]^{-1}]]$. Therefore $E < \epsilon$ unless $\| t \|$ is in I and y is in K'.

Fix a $| t |$ in I and a y in K'. Then

$$E = [d(g)]^{1/2} \left| \int_{G/H} f(s \cdot g) f(s)\, \beta^*(p(s)\, n\gamma(y)\, p(t)\, [p(s \cdot g)]^{-1}) \right.$$
$$\left. \times\, (V^*_{\pi(p(s)n\gamma(y)p(t)[p(s\cdot g)]^{-1})}\psi, \psi)\, ds \right|$$
$$= [d(n\gamma(y)p(t)]^{1/2} \left| \int_{G/H} \beta(p(s)n[p(s)]^{-1})\, f(s \cdot \gamma(y) \cdot p(t))\, f(s) \right.$$
$$\times\, \beta^*(p(s)\, \gamma(y)\, p(t)\, [p(s \cdot n \cdot \gamma(y) \cdot p(t))]^{-1})$$
$$\times\, (V^*_{\pi(p(s)n[p(s)]^{-1})}$$
$$\left. \times\, V^*_{\pi(p(s)\gamma(y)p(t)[p(s\cdot n\cdot\gamma(y)\cdot p(t))]^{-1})}\psi, \psi)\, ds \right|$$
$$= \left| \int_{G/H} [\beta^{p(s)}]\, (n) F_{(t,y)}(s)\, ds \right|,$$

where

$$F_{(t,y)}(s) = [d(\gamma(y)\, p(t))]^{1/2} f(s \cdot \gamma(y) \cdot p(t))\, f(s)$$
$$\times\, \beta^*(p(s)\, \gamma(y)\, p(t)\, [p(s \cdot \gamma(y) \cdot p(t))]^{-1})$$
$$(V^*_{\pi(p(s)\gamma(y)p(t)[p(s\cdot\gamma(y)\cdot p(t))]^{-1})}\psi, \psi).$$

($d(n) = 1$ and $s \cdot n = s$ whenever n is in N.)

The mapping $s \to \beta^{p(s)}$ is an analytic diffeomorphism of G/H onto the orbit of $\hat{N}$ containing β. Since the kernel of U^T is precisely the set of all n in N such that $\beta^{p(s)}(n) = 1$ for all s, we may consider the quotient of G by this kernel in which case the orbit of β is a Riemann–Lebesgue subset of $\hat{N}$ (see [4]). Therefore, for each pair (t, y) there exists a compact subset $Q_{(t,y)}$ of N such that $| \int_{G/H} \beta^{p(s)}(n) F_{(t,y)}(s)\, ds | < \epsilon$ whenever n is outside $Q_{(t,y)}$. (One has only to check that $F_{(t,y)}$ is integrable.) Now finally, as (t, y) varies over the compact set $I \times K'$, the functions $[F_{(t,y)}]$ vary over a compact subset of $L^1(G/H)$. So there exists a common compact set Q in N such that $| \int_{G/H} \beta^{p(s)}(n) F_{(t,y)}(s)\, ds | < \epsilon$ for all t in I, y in K', and n outside Q.

We have thus shown that $E < \epsilon$ unless $\| t \|$ is in I, y is in K' and n is in Q. This completes the proof.

Remark. It seems likely to us that this lemma holds true in much greater generality. In fact we do not know of an example violating the conjecture that: If T vanishes at infinity, then U^T vanishes at infinity.

LEMMA E. *Let M' be a connected, two-step, nilpotent Lie group with a one-dimensional torus T for its center. Let J denote the identity character of T, Then:*

(i) *Up to equivalence, there is a unique irreducible unitary representation W of M' whose restriction to T is a multiple of J.*

(ii) *The representation* W *of* (i) *vanishes at infinity on* M'.

(iii) *The induced representation* $\text{ind}_T^{M'} J$ *is a multiple of* W.

(iv) *If* α *is any automorphism of* M' *which leaves the center* Y *pointwise invariant, then the representation* $W \cdot \alpha$ *is equivalent to* W.

Proof. If z is a generator for the Lie algebra for T, then one shows easily by induction that there exists a finite sequence $[(y_i, x_i)]$ of elements of the Lie algebra of M' with the following properties:

(a) $[x_i, y_i] = z$ for all i.

(b) $[x_i, x_j] = 0$ for all i and j.

(c) $[y_i, y_j] = 0$ for all i and j.

(d) $[z, y_1, x_1, \ldots, y_j, x_j]$ is a basis for the algebra.

Hence M' is a Heisenberg group (modulo a discrete central subgroup). Now (i) is well known. One could also see this by applying the Mackey machine to the normal subgroup whose Lie algebra is spanned by z and the $[y_i]$.

(ii) follows by direct verification, since the irreducible representations of a Heisenberg group are easily constructed. (iii) follows since $[\text{ind}_T^{M'} J]\,|_T$ is, by inspection, a multiple of J, so that $\text{ind}_T^{M'} J$ is concentrated on the single point W. (iv) follows immediately from (i).

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